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UNIFORM CONVEXITY PROPERTIES FOR HYPERGEOMETRIC FUNCTIONS

NAK EUN CHO, SOON YOUNG WOO AND SHIGEYOSHI OWA

ABSTRACT. The purpose of the present paper is to give a sufficient condition for a (Gaussian) hypergeometric function to be uniformly convex of order α which is also necessary condition under additional restrictions. Similar results for the corresponding subclasses of starlike functions are also obtained. Furthermore, we examine an integral operator related to the hypergeometric function.

1. Introduction

Let $\mathcal{A}$ be the class consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of univalent, starlike and convex functions of order α, respectively. For convenience, we write $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ (see, e.g., Srivastava and Owa [11]).

Motivated by geometric considerations, Goodman [3,4] introduced the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and starlike functions. Ronning [7](also, see [5]) gave a more applicable one variable analytic characterization for $\mathcal{UCV}$. That is, a function f of the form (1.1) is in $\mathcal{UCV}$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}).$$

We note [3] that the classical Alexander's result, $f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*$, does not hold between the classes $\mathcal{UCV}$ and $\mathcal{UST}$. On later, Ronning [8] introduced the class $\mathcal{S}_p$ consisting of functions such that $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{S}_p$. And also in [7], Ronning

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX generalized the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ by introducing a parameter α in the following way.

Definition. A function f of the form (1.1) is in $\mathcal{S}_p(\alpha)$ if it satisfies the analytic characterization

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (\alpha \in \mathbb{R}; z \in \mathcal{U}).$$

and $f \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf' \in \mathcal{S}_p(\alpha)$.

For the class $\mathcal{S}_p(\alpha)$, we get a domain whose boundary is a parabola with vertex $\omega = (1+\alpha)/2$. Also, we note that $\mathcal{S}_p(\alpha) \subset \mathcal{S}^*$ for all $-1 \leq \alpha < 1$, $\mathcal{S}_p(\alpha) \not\subset \mathcal{S}$ for $\alpha < -1$ and $\mathcal{UCV}(\alpha) \subset \mathcal{K}$ for $\alpha \geq -1$.

We denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.2}$$

and let $\mathcal{UCT}(\alpha) = \mathcal{UCV}(\alpha) \cap \mathcal{T}$ and $\mathcal{S}_p\mathcal{T}(\alpha) = \mathcal{S}_p(\alpha) \cap \mathcal{T}$.

Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where $c \neq 0, -1, -2, \cdots$ and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases}$$

We note that $F(a, b; c; 1)$ converges for $\mathrm{Re}(c-a-b) > 0$ and is related to the Gamma functions by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.3}$$

Merkes and Scott [6] and Ruscheweyh and Singh [9] used continued fractions to find sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$ for various choices of the parameters a, b and c. Carlson and Shaffer [2] showed how some convolution results about $\mathcal{S}^*(\alpha)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [10] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

In the present paper, we determine sufficient conditions for $zF(a,b;c;z)$ to be in $\mathcal{S}_p(\alpha)$ and $\mathcal{UCV}(\alpha)$ and also give necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $\mathcal{S}_p\mathcal{T}(\alpha)$ and $\mathcal{UCT}(\alpha)$ with appropriate restrictions on a, b and c. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Conditions for uniform convexity

To establish our main results, we need the following lemmas due to Bharati et al.[1].

Lemma 2.1. *A sufficient condition for f of the form (1.1) to be in $\mathcal{S}_p(\alpha)$ $(-1 \leq \alpha < 1)$ is that*

$$\sum_{n=2}^{\infty}(2n-1-\alpha)|a_n| \leq 1-\alpha, \tag{2.1}$$

and a necessary and sufficient condition for f of the form (1.2) to be in $\mathcal{S}_p\mathcal{T}(\alpha)$ is that the condition (2.1) is satisfied.

Lemma 2.2. *A sufficient condition for f of the form (1.1) to be in $\mathcal{UCV}(\alpha)$ $(-1 \leq \alpha < 1)$ is that*

$$\sum_{n=2}^{\infty} n(2n-1-\alpha)|a_n| \leq 1-\alpha, \tag{2.2}$$

and a necessary and sufficient condition for f of the form (1.2) to be in $\mathcal{UCT}(\alpha)$ is that the condition (2.2) is satisfied.

By using Lemma 2.1 and Lemma 2.2, we now derive

Theorem 2.1. *If $a, b > 0$ and $c > a+b+1$, then a sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{S}_p(\alpha)$ $(-1 \leq \alpha < 1)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\frac{2ab}{(1-\alpha)(c-a-b-1)}\right) \leq 2. \tag{2.3}$$

Condition (2.3) is necessary and sufficient for F_1 defined by $F_1(a,b;c;z) = z(2 - F(a,b;c;z))$ to be in $\mathcal{S}_p\mathcal{T}(\alpha)$.

Proof. Since

$$zF(a,b;c;z) = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

according to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\alpha.$$

Now

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = 2\sum_{n=1}^{\infty}\frac{n(a)_n(b)_n}{(c)_n(1)_n} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \tag{2.4}$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (1.3), we may express (2.4) as

$$\begin{aligned}
&\frac{2ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= \frac{2ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-\alpha)\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right) \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{2ab}{c-a-b-1} + 1 - \alpha\right) - (1-\alpha).
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.3) holds. Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

the necessity of (2.3) for F_1 to be in $\mathcal{S}_p\mathcal{T}(\alpha)$ follows from Lemma 2.1.

Theorem 2.2. *If $a, b > -1$, $ab < 0$ and $c > 0$, then a necessary and sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{S}_p\mathcal{T}(\alpha)$ $(-1 \leq \alpha < 1)$ is that*

$$c \geq a+b+1-\frac{2ab}{1-\alpha}. \tag{2.5}$$

Proof. Since

$$\begin{aligned}
zF(a,b;c;z) &= z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \\
&= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,
\end{aligned} \tag{2.6}$$

according to Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha) \tag{2.7}$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2(n+2)-1-\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&= 2\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= 2\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-\alpha)\frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right).
\end{aligned}$$

Hence (2.7) is equivalent to

$$\begin{aligned}
&\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left(2+(1-\alpha)\frac{c-a-b-1}{ab}\right) \\
&\qquad \le (1-\alpha)\left(\frac{c}{|ab|}+\frac{c}{ab}\right) = 0.
\end{aligned} \tag{2.8}$$

Thus (2.8) is valid if and only if $2+(1-\alpha)(c-a-b-1)/(ab) \le 0$ or, equivalently, $c \ge a+b+1-2ab/(1-\alpha)$.

Our next two theorems will parallel Theorem 2.1 and Theorem 2.2 for the uniformly convex case.

Theorem 2.3. *If $a,b>0$ and $c>a+b+2$, then a sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{UCV}(\alpha)$ $(-1\le\alpha<1)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{2(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2} + \left(\frac{5-\alpha}{1-\alpha}\right)\left(\frac{ab}{c-a-b-1}\right)+1\right) \le 2. \tag{2.9}$$

Condition (2.9) is necessary and sufficient for $F_1(a,b;c;z) = z(2-F(a,b;c;z))$ to be in $\mathcal{UCT}(\alpha)$.

Proof. In view of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n(2n-1-\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)(2(n+2)-1-\alpha)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= 2\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - (1+\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.
\end{aligned} \tag{2.10}$$

Writing $n+2=(n+1)+1$, we have

$$\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \tag{2.11}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}&(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\&=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\&=\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.\end{aligned} \tag{2.12}$$

Substituting (2.11) and (2.12) into the right side of (2.9), we obtain

$$2\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}}+(5-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \tag{2.13}$$

Since $(a)_{n+k}=(a)_k(a+k)_n$, we write (2.13) as

$$\begin{aligned}\frac{2(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}&+(5-\alpha)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\+(1-\alpha)&\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right).\end{aligned}$$

By a simplification, we see that the last expression is bounded above by $1-\alpha$ if and only if (2.9) holds. That (2.9) is necessary for F_1 to be in $\mathcal{UCT}(\alpha)$ follows from Lemma 2.2.

Theorem 2.4. *If $a,b>-1$, $ab<0$ and $c>a+b+2$, then a necessary and sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{UCT}(\alpha)$ is that*

$$2(a)_2(b)_2+(5-\alpha)ab(c-a-b-2)+(1-\alpha)(c-a-b-1)\geq 0 \tag{2.14}$$

Proof. Since zF has the form (2.6), we see from Lemma 2.2 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(2n-1-\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|}(1-\alpha).$$

Writing $(n+2)(2(n+2)-1-\alpha) = 2(n+1)^2+(3-\alpha)(n+1)+(1-\alpha)$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)(2(n+2)-1-\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&= 2\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (3-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\quad + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&= \frac{2(a+1)(b+1)}{c+1}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + (5-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\quad + (1-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big(2(a+1)(b+1)+(5-\alpha)(c-a-b-2) \\
&\quad + \frac{1-\alpha}{ab}(c-a-b-1)_2\Big) - \frac{(1-\alpha)c}{ab}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)c/|ab|$ if and only if

$$2(a+1)(b+1)+(5-\alpha)(c-a-b-2)+\frac{1-\alpha}{ab}(c-a-b-1)_2 \leq 0,$$

which is equivalent to (2.14).

3. An integral operator

In this section, we obtain similar type results in connection with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z) = \int_0^z F(a,b;c;t)dt. \tag{3.1}$$

Theorem 3.1. (i) *If $a,b>1$ and $c>a+b-1$, then a sufficient condition for $G(a,b;c;z)$ defined by (3.1) to be in $\mathcal{S}_p(\alpha)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(2-\frac{(1+\alpha)(c-a-b)}{(a-1)(b-1)}\right)+\frac{(1+\alpha)(c-1)}{(a-1)(b-1)}\leq 2(1-\alpha), \qquad (3.2)$$

(ii) *If* $a, b > -1$, $ab < 0$ *and* $c > \max\{0, a+b+1\}$, *then* $G(a,b;c;z)$ *defined by (3.1) is in* $S_p\mathcal{T}(\alpha)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{2}{ab}-\frac{(1+\alpha)(c-a-b-1)_2}{(a-1)_2(b-1)_2}\right)+\frac{(1+\alpha)(c-1)_2}{(a-1)_2(b-1)_2}\leq 0. \qquad (3.3)$$

Proof. Since

$$G(a,b;c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n,$$

we note that

$$\begin{aligned}&\sum_{n=2}^{\infty}(2n-1-\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\\ &=2\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-(1+\alpha)\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}-1\right)\\ &=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(2-\frac{(1+\alpha)(c-a-b)}{(a-1)(b-1)}\right)+\frac{(1+\alpha)(c-1)}{(a-1)(b-1)}-(1-\alpha).\end{aligned}$$

which is bounded above by $1-\alpha$ if and only if (3.2) holds. This completes the proof of (i). To prove (ii), we apply Lemma 2.1 to

$$G(a,b;c;z)=z-\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n.$$

It suffices to show that

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\leq(1-\alpha)\frac{c}{|ab|}.$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2(n+2)-1-\alpha)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}\\
&=2\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}-(1+\alpha)\frac{c}{ab}\left(\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}-1\right)\\
&=\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{2}{ab}-\frac{(1+\alpha)(c-a-b-1)_2}{(a-1)_2(b-1)_2}\right)\\
&+\frac{(1+\alpha)(c-1)_2}{(a-1)_2(b-1)_2}-(1-\alpha)\frac{c}{ab}\\
&\leq(1-\alpha)\frac{c}{|ab|},
\end{aligned}$$

which is equivalent to (3.3).

Now we observe that $G(a,b;c;z)\in\mathcal{UCV}(\alpha)(\mathcal{UCT}(\alpha))$ if and only if $zF(a,b;c;z)\in\mathcal{S}_p(\alpha)(\mathcal{S}_p\mathcal{T}(\alpha))$. Thus any result of functions belonging to the class $\mathcal{S}_p(\alpha)(\mathcal{S}_p\mathcal{T}(\alpha))$ about zF leads to that of functions belonging to the class $\mathcal{UCV}(\alpha)(\mathcal{UCT}(\alpha))$. Hence we obtain the following analogs to Theorem 2.1 and Theorem 2.2.

Theorem 3.2. (i) *If $a,b>0$ and $c>a+b+1$, then a sufficient condition for $G(a,b;c;z)$ defined by (3.1) to be in $\mathcal{UCV}(\alpha)$ $(-1\leq\alpha<1)$ is that the inequality (2.3) is satisfied.*

(ii) *If $a,b>-1$, $ab<0$ and $c>a+b+2$, then $G(a,b;c;z)$ defined by (3.1) is in $\mathcal{UCT}(\alpha)$ $(-1\leq\alpha<1)$ if and only if the enequality (2.5) is satisfied.*

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Nak Eun Cho and Soon Young Woo
Department of Applied Mathematics
Pukyong National University
Pusan 608-737, Korea.

Shigeyoshi Owa
Dcpartmcnt od Mathcmatics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan